↳ Equivalence relation's impact on natural mapping

- Given a set *A* and an equivalence relation ***R* on *A***, a ***natural mapping g*** : *A*→*A*/*R* can be determined.
- Natural mappings vary with the equivalence relation:
  - The **identity relation** yields a **bijection.**
  - Others equivalence relations are generally **surjections** only.
- **Example**: Let $A=\{1, 2, 3\}$,
  - **Equivalence relation** : $R_1=\{<1,2>,<2,1>\}\cup I_A$

    Natural mapping : $g_1(1) = g_1(2) = \{1,2\}$, $g_1(3) = \{3\}$
  - **Equivalence relation**: I_A

    Natural mapping: $g_2(1)=\{1\}$, $g_2(2)=\{2\}$, $g_2(3)=\{3\}$

- 5.1.1 Definition of a Function
- 5.1.2 Image and Preimage of a Function

- **Definition 5.7**: Let f:$A \to B$ be a function, and let $A_1 \subseteq A$, $B_1 \subseteq B$. Then:
  - $f(A_1)=\{f(x) \mid x \in A_1\}$ is called the **image of A_1 under f**. In particular, $f(A)$ is called the **image of the function.**
  - $f^{-1}(B_1)=\{x \mid x \in A \wedge f(x) \in B_1\}$ is called the **complete preimage of B_1 under f.**
- **Note :**
  - A function value $f(x) \in B$ is a point-to-point result, while an image $f(A_1) \subseteq B$ is a set-to-set transformation.
  - $A_1 \subseteq f^{-1}(f(A_1))$:The preimage of the image of A_1 may contain more elements than A_1 itself.
  - $f(f^{-1}(B_1)) \subseteq B_1$:Taking the image of the preimage of B_1 may not recover the original set B_1.
  - A **complete preimage** is the source of a set of outputs; a **preimage** refers to the source of a single output. Both follow the same set operation rules.

■ **Examples:**

(1) Let f : N→N, and let $f(x)=\begin{cases}\frac{x}{2} & if\ x\ even\\ x+1 & if\ x\ odd\end{cases}$

$A=\{0,1\}$, $B=\{2\}$, then

$f(A) = f(\{0,1\}) = \{ f(0), f(1) \}=\{ 0, 2 \}$

$f(B) = \{ f(2) \} = \{ 1 \}$

(2) $A=\{1, 2, 3\}$, $B=\{a, b, c\}$, $f=\{<1,a>,<2,a>,<3,b>\}$, then

- $f^{-1}(\{a,b\}) =\{1,2,3\}$, $f^{-1}(\{b,c\}) =\{3\}$
- $\{1\}\subset\{1,2\}=f^{-1}(\{a\})= f^{-1}(f(\{1\}))$ （Non-injective functions expand preimages）
- $f(f^{-1}(\{b,c\}))=f(\{3\})=\{b\}\subset\{b, c\}$ （Non-surjective ones shrink images ）

- 5.1.1 Definition of a Function
- 5.1.2 Image and Preimage of a Function
- 5.1.3 Properties of a Function
  - Surjective, Injective, and Bijective Functions
  - Constructing a bijective function

Definitions of Surjective, Injective, and Bijective Functions

Note:

- **Surjectivity** means: for $\forall y \in B$, here exists an $x \in A$ such that $f(x)=y$.
- **Injectivity** means: if ： $f(x_1)=f(x_2) \Rightarrow x_1=x_2$.
- A surjection ensures full coverage of the codomain, an injection ensures no duplication in mapping, and a bijection guarantees reversibility.

Examples: Determine whether the following functions are injective, surjective, or bijective, and explain why.

（1） f: $\mathbf{R} \to \mathbf{R}$, $f(x)=-x^2+2x-1$

Solution: When x=0,2， $f(x)$=-1， so it is not injective .

- **Examples:** Determine whether the following functions are injective, surjective, or bijective, and explain why.

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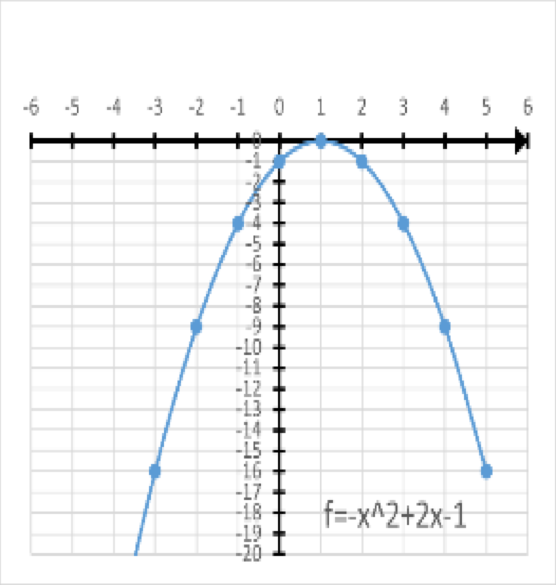
Solution: When x=0,2， $f(x)$=-1， so it is ***not injective*** .

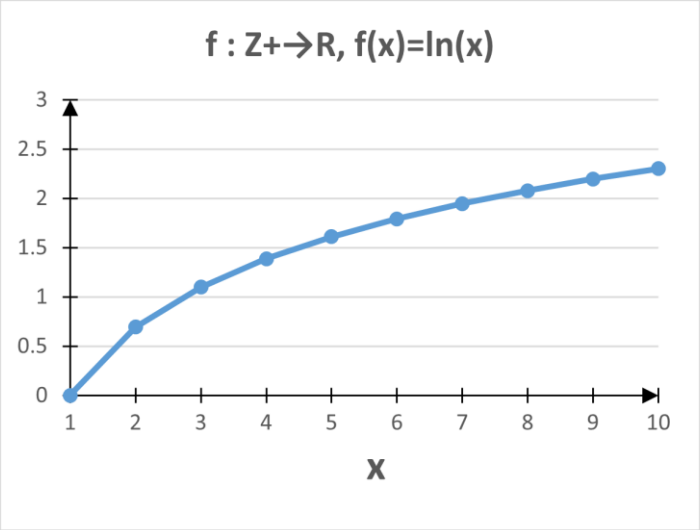


As shown in the figure, the function ***f*** cannot map to any positive real number, and thus it is ***neither surjective nor bijective***.

■ **Examples:** Determine whether the following functions are injective, surjective, or bijective, and explain why.

（2） *f*： $Z^+ \rightarrow R$, f(x)=lnx, Z^+ is the set of positive integers.

Solution:

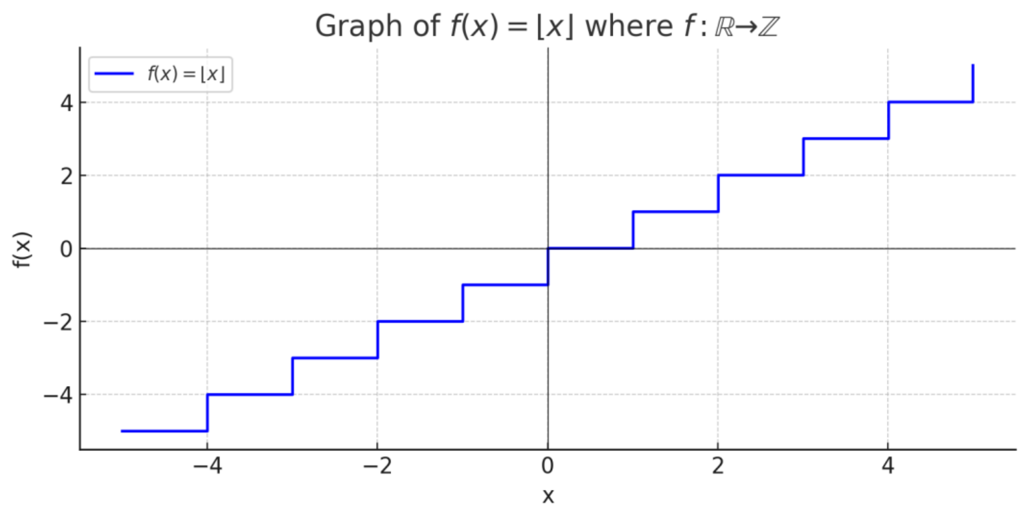


Solution: ***f*(*x*)** is monotonically increasing, so it is ***injective***.
Since **ran*f*={ln1,ln2,…}** cannot cover all values in the real number set **R**, ***f*** is not surjective.

■ **Examples:** Determine whether the following functions are injective, surjective, or bijective, and explain why.

（3） $f: R \to Z, f(x) = \lfloor x \rfloor$

Solution:

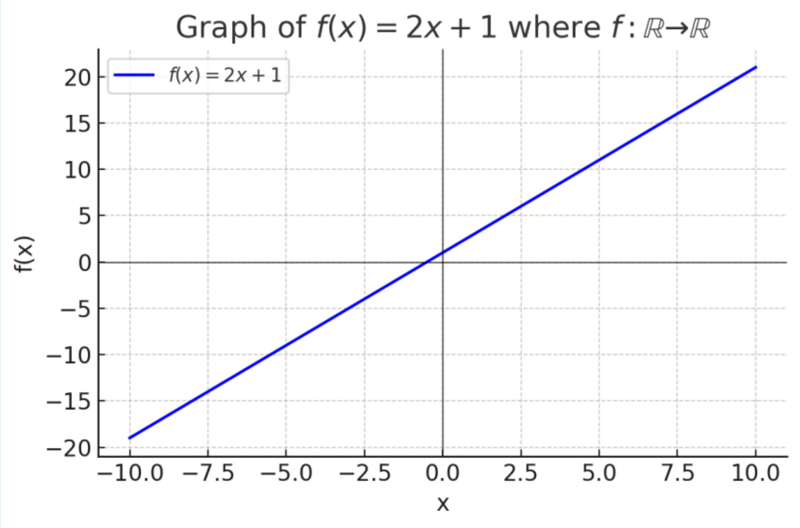


Every integer ***f*(*x*)** has a corresponding real number ***x***, so the function is ***surjective***. However, different real numbers may have the same floor value ***f*(*x*)**, so the function is ***not injective***.

- **Examples:** Determine whether the following functions are injective, surjective, or bijective, and explain why.

（4） $f:\mathrm{R}\rightarrow\mathrm{R},\ f(x)=2x+1$

Solution:

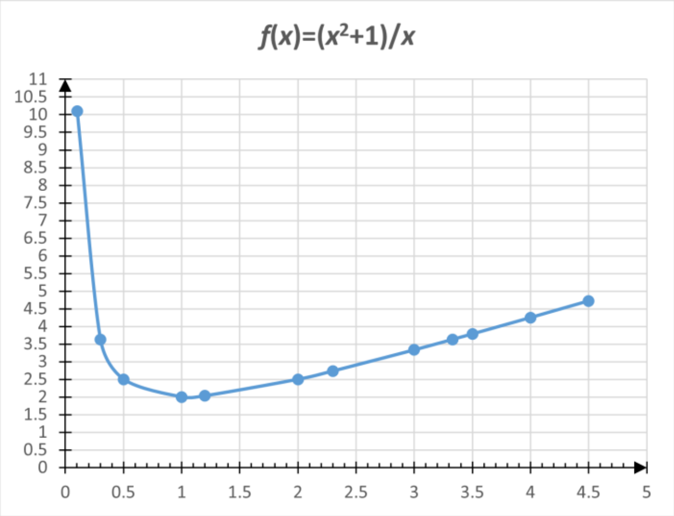


Surjective, ***injective***, ***bijective***, because it is monotonic and the range of ran$f$=R

- **Examples:** Determine whether the following functions are injective, surjective, or bijective, and explain why.

（5） $f: R^+ \to R^+, f(x)=(x^2+1)/x$, where R^+ is the set of positive real numbers.

Solution:



The function has a minimum value $f(1)=2$. This function is ***neither injective nor surjective***.

- Example: $A=P(\{1,2,3\})$, $B=\{0,1\}^{\{1,2,3\}}$

  Solve: $A=\{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

  $B=\{f_0, f_1, \ldots , f_7\}$.

  $f_0=\{<1,0>,<2,0>,<3,0>\}$, $f_1=\{<1,0>,<2,0>,<3,1>\}$,
  $f_2=\{<1,0>,<2,1>,<3,0>\}$, $f_3=\{<1,0>,<2,1>,<3,1>\}$,
  $f_4=\{<1,1>,<2,0>,<3,0>\}$, $f_5=\{<1,1>,<2,0>,<3,1>\}$,
  $f_6=\{<1,1>,<2,1>,<3,0>\}$, $f_7=\{<1,1>,<2,1>,<3,1>\}$.

  Let $f : A\to B$,

  $f(\varnothing)=f_0$, $f(\{1\})=f_1$, $f(\{2\})=f_2$, $f(\{3\})=f_3$,
  $f(\{1,2\})=f_4$, $f(\{1,3\})=f_5$, $f(\{2,3\})=f_6$, $f(\{1,2,3\})=f_7$

- Each subset in A is mapped to its characteristic function. For example, $f(\{2\})=f_2$, since only element 2 is in the subset, its characteristic value is (0,1,0).

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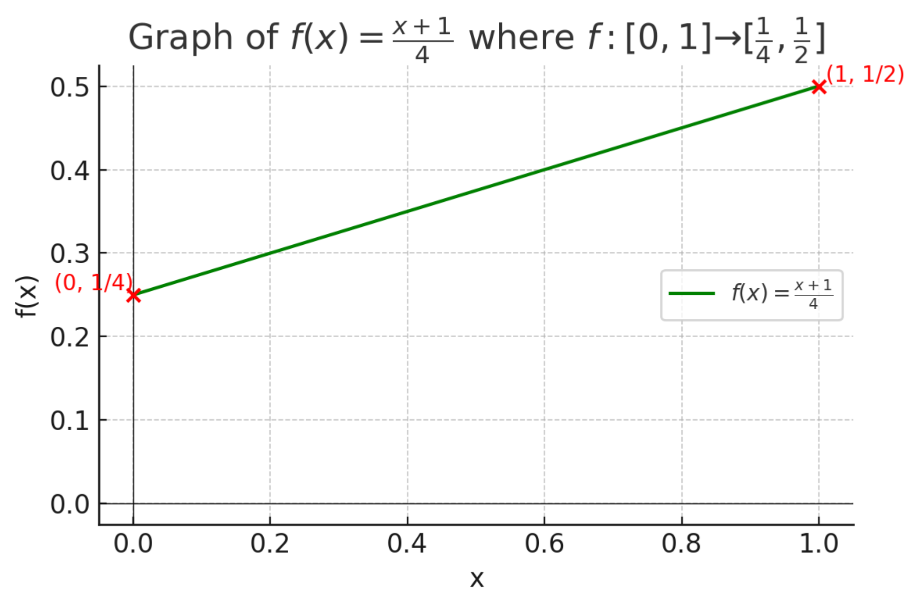
- **Construction Method: Linear Equation**
- **Example:** $A=[0,1]$, $B=[1/4,1/2]$

  Construct a bijection $f : A \to B$

Solve: To map $A=[0,1]$ onto $B=[1/4,1/2]$, match the endpoints and use a straight-line function to create a bijection.

Let $f : [0,1] \to [1/4,1/2]$

$$f(x)=(x+1)/4$$



- **Construction Method:** Arrange the elements of set *A* in a specific order based on a certain criterion. Then, starting from the first element, map them sequentially to the natural numbers.
- **Example:** A=Z, B=N， Construct a bijection $f : A \to B$

Solve : Arrange the elements of Z in the following order and correspond them with the elements of N:

$$\begin{array}{cccccccc} \text{Z:} & 0 & -1 & 1 & -2 & 2 & -3 & 3 \ldots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{N:} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \ldots \end{array}$$

The function represented by this correspondence is:

$$f: \mathrm{Z} \to \mathrm{N}, f(x) = \begin{cases} 2x & x \geq 0 \\ -2x-1 & x < 0 \end{cases}$$

Objective :

Key Concepts ：

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5.2.1 Composition of Functions

- The fundamental theorem of function composition and its corollaries
- Properties of function composition

5.2.2 Inverse Functions

- Conditions for the existence of an inverse function
- Properties of inverse function

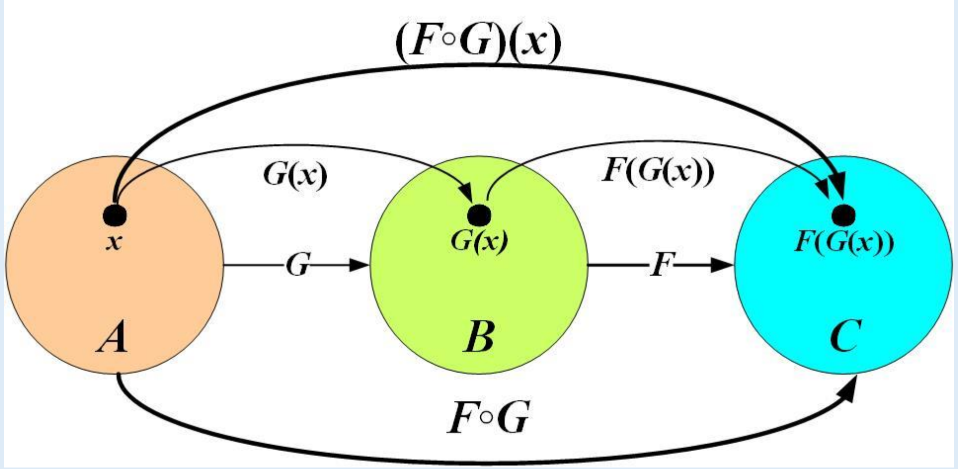
- **Theorem 5.1**: Let F,G be functions, **then the composition** $F\circ G$ is also a function and satisfies the following conditions:

  (1) **dom($F\circ G$)={ x | $x\in$domG $\wedge$ $G(x)\in$domF}**

  (This describes the relationship between the domains and ranges of the functions.)

  (2) **$\forall x\in$dom($F\circ G$), $F\circ G(x) = F(G(x))$**

  (This specifies the order of computation for the composite function.)



Note: The composition of functions is **left composition** with **right-hand priority**, while the composition of relations is **right composition** with **left-hand priority**.

- **Theorem 5.2:** Let $f : B \rightarrow C$, $g : A \rightarrow B$.

  (1) If f , g are surjective, then $f \circ g : A \rightarrow C$ is also ***surjective***.

  (2) If both f , g are injective, then the composition $f \circ g : A \rightarrow C$ is also ***injective***.

  (3) If both f , g are bijective, then the composition $f \circ g : A \rightarrow C$ is also ***bijective***.

- **Theorem 5.2:** Let $f : B \to C$, $g : A \to B$.

(1) If f , g are surjective, then $f \circ g : A \to C$ is also ***surjective***.

proof : **Goal:** Prove that for any $c \in C$, there exists at least one element $a \in A$ such that $(f \circ g)(a) = c$.

①Since f is surjective, for any $c \in C$, there exists some $b \in B$ such that $f(b) = c$.

②Since g is also surjective, there exists some $a \in A$ such that $g(a) = b$.

③Given that $g(a) = b$ and $f(b) = c$, by the definition of function composition, we have: $(f \circ g)(a) = f(g(a)) = f(b) = c$.

Therefore, $f \circ g : A \to C$ is ***surjective***.

■ **Theorem 5.2:** Let $f : B \to C$, $g : A \to B$.

(2) If both f , g are injective, then the composition $f \circ g : A \to C$ is also *injective*.

proof : **Goal:** We need to prove that if $x_1, x_2 \in A$, $x_1 \neq x_2$, *then* $f \circ g(x_1) \neq f \circ g(x_2)$.

①Since g injective, $g(x_1) \neq g(x_2)$, and $g(x_1)$, $g(x_2) \in B = \mathrm{dom} f$。

②*Since* f injective, we know that $f(g(x_1)) \neq f(g(x_2))$, thus $f \circ g(x_1) \neq f \circ g(x_2)$.

so $f \circ g : A \to C$ *is injective.*

- 5.2.1 Composition of Functions
  - The fundamental theorem of function composition and its corollaries
  - Properties of function composition
- 5.2.2 Inverse Functions
  - Conditions for the existence of an inverse function
  - Properties of inverse function

- **Theorem 5.4: Let $f: A \to B$ *be bijective*, so $f^{-1}: B \to A$ *is also bijective*.**

Proof:

①Since f is a function, so f^{-1} is relation, and we have **domf^{-1}= ran$f$ =B** , **ranf^{-1}= dom$f$ =A.**

②For any $x \in B$, suppose have $y_1, y_2 \in A$ such that $<y_1,x> \in f^{-1} \wedge <y_2, x> \in f^{-1}$, then by the definition of inverse $<x,y_1> \in f \wedge <x,y_2> \in f$. Since f is injective, it follows that $y_1=y_2$, **hence f^{-1} is *a well-defined function*.**

③Also, for every $x \in B$, there is a unique $a \in A$ such that $f(a)=x$, so $f^{-1}(x)=a$, therefore, f^{-1} ***is surjective***.

④Now, suppose exist $x_1, x_2 \in B$ such that $f^{-1}(x_1) = f^{-1}(x_2)=y$, then we have $<y, x_1> \in f^{-1} \wedge <y, x_2> \in f^{-1} \Rightarrow <x_1,y> \in f \wedge <x_2,y> \in f \Rightarrow x_1=x_2$ (Since f is injective *function*),then proves that f^{-1} **is *injective*** .

Conclusion: Since f^{-1} is both **injective** and **surjective**, $f^{-1}:B \to A$ is a ***bijective function.***

- **Example: Let** $f: R \rightarrow R$, $g: R \rightarrow R$, $f(x) = \begin{cases} x^2 & x \geq 3 \\ -2 & x < 3 \end{cases}$, $g(x) = x + 2$
  - Find $f \circ g$, $g \circ f$.
  - If f and g have inverse functions, find their inverses.

Solve :

$g \circ f$: $\mathbf{R} \rightarrow \mathbf{R}$ $\quad$ $f \circ g$: $\mathbf{R} \rightarrow \mathbf{R}$

$$g \circ f(x) = \begin{cases} x^2 + 2 & x \geq 3 \\ 0 & x < 3 \end{cases} \qquad f \circ g(x) = \begin{cases} (x+2)^2 & x \geq 1 \\ -2 & x < 1 \end{cases}$$

$f: R \rightarrow R$ does not have an inverse function .

$g: R \rightarrow R$ inverse function is $g^{-1}: R \rightarrow R$, $g^{-1}(x) = x - 2$.

↳ Identity Composition Theorem of Inverse Functions

- **The composition of a bijective function and its inverse is the identity function.**
- **Theorem 5.5: Let** $f : A \to B$ be a **bijective** function, then:

$$f^{-1} \circ f = I_A, \quad f \circ f^{-1} = I_B$$

Proof:

① According to Theorem **5.4** $f^{-1}: B \to A$ is also bijective.

② By the theorem on function composition, $f^{-1} \circ f : A \to A$, $f \circ f^{-1}: B \to B$, and both equal the corresponding **identity functions.**

③ By the above Therefore, we have: $f^{-1} \circ f = I_A$, $f \circ f^{-1} = I_B$.

- The left and right compositions of a bijective function and its inverse are identity functions on their respective sets, and I_A is generally not equal to I_B .

Objective :

Key Concepts ：